Versions of the circle method

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What is the circle method?

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- Generating function for sequence $(a_n)_{n=0}^{\infty}$: $\sum_{n=0}^{\infty} a_n q^n$

Example (Generating function for $(1, 1, 1, \ldots)$)

The generating function for the sequence $(1, 1, 1, \ldots)$ is

$$1 + q + q^2 + q^3 + \cdots = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

if $|q| < 1$.

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- A collection of techniques for using the analytic properties of the generating function of a sequence to obtain an asymptotic formula for the sequence
- Typically refers to the Hardy–Littlewood circle method
- May also refer to other methods that are used to provide an asymptotic formula for the number of ways an integer is represented by an integer-valued function (like a quadratic form) on $\mathbb{Z}^s$
  - Kloosterman circle method
  - The delta(-symbol) method

Partitions of a positive integer

Definition (Partition of a positive integer)

Let n be a positive integer. A **partition** of n is a way to write n as the sum of positive integers, where the order of the summands **does not** matter.
The **partition function** $p(n)$ is the number of partitions of n.

Example (Partitions of 5)

$$5 \qquad 2+2+1$$
$$4+1 \qquad 2+1+1+1$$
$$3+2 \qquad 1+1+1+1+1$$
$$3+1+1$$

$\implies p(5) = 7$

Hardy–Littlewood circle method

- Originally developed by Hardy and Ramanujan (1918) to provide asymptotic formula for the partition function $p(n)$
- Proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n) \mathrm{e}(nz) \,,$$

where $\mathrm{e}(z) = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$.

Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz)\,,$$

where $\mathrm{e}(z) = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$.

$$f(z) = \frac{\mathrm{e}(z/24)}{\eta(z)},$$

where $\eta(z)$ is the Dedekind eta function

$$\eta(z) = \mathrm{e}\left(\frac{z}{24}\right) \prod_{m=1}^{\infty} (1 - \mathrm{e}(mz)).$$

Hardy and Ramanujan used the modularity of η to obtain the asymptotic formula.

Hardy–Littlewood circle method & partition function

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz) \qquad F(q) = \sum_{n=0}^{\infty} p(n)q^n$$

where $q = \mathrm{e}(z)$.

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Using the Cauchy integral formula, we find that

$$p(n) = \frac{1}{2\pi i}\int_{|q|=r} \frac{F(q)}{q^{n+1}}\, dq,$$

where $0 < r < 1$.

Hardy–Littlewood circle method & partition function

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Using the Cauchy integral formula, we find that

$$p(n) = \frac{1}{2\pi i}\int_{|q|=r} \frac{F(q)}{q^{n+1}}\, dq,$$

where $0 < r < 1$.
Changing q into $\mathrm{e}(x+iy)$, we obtain

$$p(n) = \int_0^1 f(x+iy)\mathrm{e}(-n(x+iy))\, dx,$$

where $y > 0$ is such that $r = e^{-2\pi y}$.

Partition function

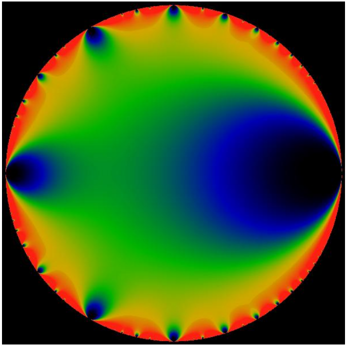

Figure: Modulus of $\prod_{m=1}^{\infty}(1 - q^m)$ with $|q| < 1$. From Wikipedia.

Main contribution to integral from points near $\mathrm{e}(a/q)$ where q is small.

Major arcs and minor arcs

Split $[0, 1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

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$$\mathfrak{M} = \left\{ x \in [0,1] : x \text{ is ``close to''} \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$

How close depends on the application of the method.

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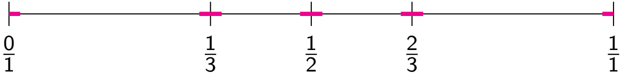
How close depends on the application of the method.

$$\mathfrak{m} = [0,1] \setminus \mathfrak{M}.$$

Example of major arcs $\mathfrak{M}$ when $Q = 3$ for the Hardy–Littlewood circle method:



$$
\begin{aligned}
p(n) &= \int_0^1 f(x+iy)\mathrm{e}(-n(x+iy))\ dx \\
&= \underbrace{\int_{\mathfrak{M}} f(x+iy)\mathrm{e}(-n(x+iy))\ dx}_{\text{main term}} + \underbrace{\int_{\mathfrak{m}} f(x+iy)\mathrm{e}(-n(x+iy))\ dx}_{\text{error term}}
\end{aligned}
$$

Real quadratic forms

F is a real quadratic form in s variables $\iff$
For all $\mathbf{m} \in \mathbb{R}^s$,

$$F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m},$$

where A is a real symmetric $s \times s$ matrix and is the Hessian matrix of F.

Example (Example of a quadratic form in 2 variables)

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2}\mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

Quadratic form definitions

Definition (Integral quadratic form)

A quadratic form F is **integral** if $F(\mathbf{m}) \in \mathbb{Z}$ for all $\mathbf{m} \in \mathbb{Z}^s$.

Definition (Positive definite quadratic form)

A quadratic form F is **positive definite** if $F(\mathbf{m}) > 0$ for all $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$.

Examples (Examples of integral positive definite quadratic forms)

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

Hardy–Littlewood circle method & quadratic forms

Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

Want an asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.

Definition (Representation number)

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Want an asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.

Use same overall method for obtaining an asymptotic formula for the partition function.

Note that the theta function

$$\Theta(z) = \sum_{n=0}^{\infty} R_F(n)\mathrm{e}(nz)$$

is a modular form.

Singular series $\mathfrak{S}_F(n)$

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q} \left(F(\mathbf{h}) - n\right) \right)$$

The singular series $\mathfrak{S}_F(n)$ contains information about $F(\mathbf{m}) \equiv n \pmod{q}$ for all positive integers q.

$\mathfrak{S}_F(n) = 0 \iff$
there exists a positive integer q such that $F(\mathbf{m}) \equiv n \pmod{q}$ has no solutions.

An asymptotic for representation numbers from Hardy–Littlewood circle method

Theorem (Kloosterman, 1924)

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 5$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

Motivation for the Kloosterman circle method

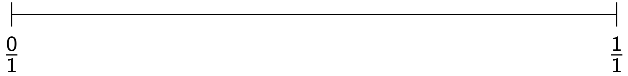
- Want a better error term in asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.
- Split $[0, 1]$ differently.

Farey sequence $\mathfrak{F}_Q$ of order Q

Definition

For $Q \geq 1$, the **Farey sequence** $\mathfrak{F}_Q$ **of order** Q is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 1$



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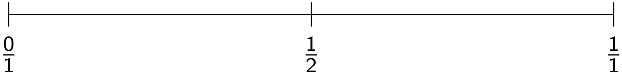
$Q = 2$



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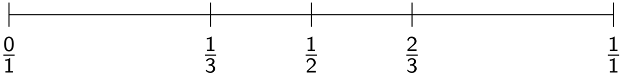
$Q = 3$



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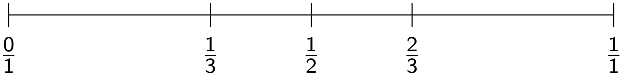
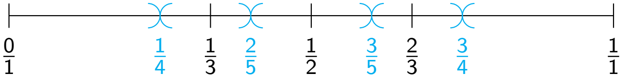


Example of Farey dissection when $Q = 3$:



An asymptotic for representation numbers from the Kloosterman method

Theorem

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

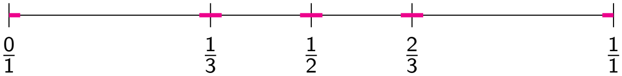
$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ($F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$), using what is now called the Kloosterman circle method.

Hardy–Littlewood vs. Kloosterman

Example of major arcs when $Q = 3$ for the Hardy–Littlewood circle method:



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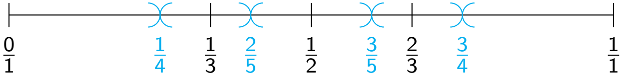

Hardy–Littlewood vs. Kloosterman

Hardy–Littlewood:

$$R_F(n) = \mathfrak{S}_F(n)\frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}}n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

Kloosterman:

$$R_F(n) = \mathfrak{S}_F(n)\frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}}n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

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- Rewrite $\delta(n)$, the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method
- Developed by Duke, Friedlander, and Iwaniec in 1993 to compute bounds for automorphic L-functions
- Has been used for a variety of applications, including
    - Asymptotic formulas for weighted representation numbers of quadratic forms (Heath-Brown)
    - Subconvexity bounds for (twists of) automorphic forms (Munshi)

(Unweighted) representation number

Definition ((Unweighted) representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

$$R_F(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where $\mathbf{1}_{\{F(\mathbf{m})=n\}}$ is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}^s$ is denoted by $C_c^\infty(\mathbb{R}^s)$. A function $\psi \in C_c^\infty(\mathbb{R}^s)$ is called a **bump function**.

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Let $\psi \in C_c^\infty(\mathbb{R}^s)$.
For $X > 0$, define

$$\psi_X(\mathbf{m}) = \psi\left(\frac{1}{X}\mathbf{m}\right).$$

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Definition (Weighted representation number)

$$R_{F,\psi,X}(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}}\psi_X(\mathbf{m})$$

Indicator function

$$\delta(n) = \mathbf{1}_{\{n=0\}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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$$\Longrightarrow \begin{cases} R_F(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \delta(F(\mathbf{m}) - n) \\ R_{F,\psi,X}(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \delta(F(\mathbf{m}) - n)\psi_X(\mathbf{m}) \end{cases}$$

The delta method & bump functions

For the delta method, we require $w \in C_c^\infty(\mathbb{R})$, $w(0) = 0$, and $\sum_{q=1}^\infty w(q) \neq 0$.

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If n is an integer, then

$$\delta(n) = \frac{1}{\sum_{q=1}^\infty w(q)} \sum_{q|n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right),$$

where the sum over $q \mid n$ is taken to be the sum over the positive divisors of n.

Using the fact that

$$\frac{1}{q} \sum_{a \pmod{q}} \mathrm{e}\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

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we have

$$\begin{aligned} \delta(n) &= \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q \mid n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right) \\ &= \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \pmod q} \mathrm{e}\left(\frac{an}{q}\right) \left(w(q) - w\left(\frac{|n|}{q}\right)\right) \end{aligned}$$

if n is an integer.

The delta method

$$\delta(n) = \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \pmod q} \mathrm{e}\left(\frac{an}{q}\right) \left(w(q) - w\left(\frac{|n|}{q}\right)\right)$$

if n is an integer.
Bump functions are easier to handle analytically than the discontinuous delta function, which helps when analyzing

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \delta(F(\mathbf{m}) - n) \quad \text{or}$$

$$R_{F,\psi,X}(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \delta(F(\mathbf{m}) - n) \psi_X(\mathbf{m}).$$

Specifics depend on the application of the delta method.

An asymptotic for weighted representation numbers

Theorem (Heath-Brown, 1996)

Suppose that n is an integer.
Suppose that F is a nonsingular integral quadratic form in $s \geq 4$ variables.
Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function.
Then for $\varepsilon > 0$, the weighted representation number $R_{F,\psi,n^{1/2}}(n)$ is

$$R_{F,\psi,n^{1/2}}(n) = \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n, n^{1/2})n^{\frac{s}{2}-1} + O_{F,\psi,s,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right),$$

where

$$\sigma_{F,\psi,\infty}(n,X) = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon} \psi(\mathbf{m})\, d\mathbf{m}.$$

Proof uses the delta method with a Kloosterman refinement.

Theorem (J., 2024)

Suppose that n is a positive integer.
Suppose that F is a nonsingular integral quadratic form in $s \geq 4$ variables.
Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function.
For $\varepsilon > 0$ and sufficiently large X, there is an asymptotic formula for $R_{F,\psi,X}(n)$ where the implicit constants only depend on ψ, s, and ε.
(Other constants dependent on the quadratic form are explicitly computed.)

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- Used the Kloosterman circle method (and not the delta method)
- If $F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m}$, then explicit constants depend on the eigenvalues of A and the smallest integer L such that $LA^{-1} \in M_s(\mathbb{Z})$.

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For $\varepsilon > 0$ and sufficiently large X, there is an asymptotic formula for $R_{F,\psi,X}(n)$ where the implicit constants only depend on ψ, s, and ε.
(Other constants dependent on the quadratic form are explicitly computed.)

Explicit constants are used in a variety of applications, including in computations.

Thank you for listening!

My main theorem

Theorem 1.1. *Suppose that n is a positive integer. Suppose that F is a nonsingular integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Let σ_1 be largest singular value of A, and let ν be the number of positive eigenvalues of A. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$. Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function. Then for $X \geq 1/\sigma_1$ and $\varepsilon > 0$, the weighted representation number $R_{F,\psi,X}(n)$ is*

$$\begin{aligned} &R_{F,\psi,X}(n) \\ &= \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2} \\ &\quad + O_{\psi,s,\varepsilon}\left(\frac{L^{s/2}X^{(s-1)/2+\varepsilon}\sigma_1^{(3-s)/2+\varepsilon}}{\Gamma(\nu/2)\left(\prod_{j=1}^{\nu}\lambda_j\right)^{1/2}} \left(\frac{n}{X^2} - \frac{\rho_\psi{}^2}{2} 1_{\{\nu>1\}} \sum_{j=\nu+1}^{s} \lambda_j \right)^{\nu/2-1} \right. \\ &\qquad\qquad \left. \times \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right) \\ &\quad + O_{\psi,s,\varepsilon}\left(X^{(s-1)/2+\varepsilon}\sigma_1^{(s+1)/2+\varepsilon}L^{s/2}\tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right), \end{aligned} \tag{1.6}$$

where $\lambda_1, \lambda_2, \dots, \lambda_\nu$ are the positive eigenvalues of A and $\lambda_{\nu+1}, \lambda_{\nu+2}, \dots, \lambda_s$ are the negative eigenvalues of A.

Corollary to my main theorem

Corollary 1.5. *Suppose that F is a nonsingular integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Let σ_1 be largest singular value of A, and let ν be the number of positive eigenvalues of A. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$. If n is a positive integer and $\varepsilon > 0$, then the weighted representation number $R_{F,\psi,X}(n)$ is*

$$R_{F,\psi,X}(n) = \mathfrak{S}_F(n)\,\sigma_{F,\psi,\infty}\left(n, n^{1/2}\right) n^{s/2-1} + O_{\psi,s,\varepsilon}\left(\left(\left(\sigma_1^{(s+1)/2+\varepsilon} + \frac{\sigma_1^{(3-s)/2+\varepsilon}}{\Gamma(\nu/2)\left(\prod_{j=1}^{\nu}\lambda_j\right)^{1/2}}\left(1 - \frac{{\rho_\psi}^2}{2}\mathbf{1}_{\{\nu>1\}}\sum_{j=\nu+1}^{s}\lambda_j\right)^{\nu/2-1}\right) \times n^{(s-1)/4+\varepsilon/2}\tau(n)L^{s/2}\prod_{p \mid 2\det(A)}(1-p^{-1/2})^{-1}\right),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ are the positive eigenvalues of A and $\lambda_{\nu+1}, \lambda_{\nu+2}, \ldots, \lambda_s$ are the negative eigenvalues of A.

Lemma for Kloosterman circle method

Lemma

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then

$$\int_0^1 f(x)\,dx = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q<d\le q+Q \\ qdx<1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\,dx \right),$$

where d^* is the multiplicative inverse of d modulo q.

Use this for

$$f(x) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}((x+iy)(F(\mathbf{m})-n)),$$

where $y > 0$.